The strong rainbow vertex-connection of graphs *

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Abstract

A vertex-colored graph G is said to be *rainbow vertex-connected* if every two vertices of G are connected by a path whose internal vertices have distinct colors, such a path is called a rainbow path. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. If for every pair u, v of distinct vertices, G contains a rainbow $u-v$ geodesic, then G is *strong rainbow vertex-connected*. The minimum number k for which there exists a k-vertex-coloring of G that results in a strongly rainbow vertex-connected graph is called the *strong rainbow vertex-connection number* of G, denoted by $srvc(G)$. Observe that $rvc(G) \leq srvc(G)$ for any nontrivial connected graph G.

In this paper, sharp upper and lower bounds of $srvc(G)$ are given for a connected graph G of order n, that is, $0 \leq srvc(G) \leq n-2$. Graphs of order n such that $srvc(G) = 1, 2, n-2$ are characterized, respectively. It is also shown that, for each pair a, b of integers with $a \geq 5$ and $b \geq (7a-8)/5$, there exists a connected graph G such that $rvc(G) = a$ and $srvc(G) = b$.

Keywords: vertex-coloring; rainbow vertex-connection; (strong) rainbow vertex-connection number.

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1 Introduction

All graphs considered in this paper are finite, undirected and simple. We follow the notation and terminology of Bondy and Murty [1], unless otherwise stated. Consider an edge-coloring (not necessarily proper) of a graph $G = (V, E)$. We say that a path of G is rainbow, if no two edges on the path have the same color. An edge-colored graph G is *rainbow connected* if every two vertices are connected by a rainbow path. An edge-coloring is a *strong rainbow coloring* if between every pair of vertices, one of their geodesics, i.e., shortest paths, is a rainbow path. The minimum number of colors required to rainbow color a graph G is called *the rainbow connection number*, denoted by $rc(G)$.

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Similarly, the minimum number of colors required to strongly rainbow color a graph G is called the *strong rainbow connection number*, denoted by $src(G)$. Observe that $rc(G) \leq src(G)$ for every nontrivial connected graph G. The notions of rainbow coloring and strong rainbow coloring were introduced by Chartrand et al. [4]. There are many results on this topic, we refer to [2, 3].

In [7], Krivelevich and Yuster proposed a similar concept, the concept of rainbow vertex-connection. A vertex-colored graph G is *rainbow vertex-connected* if every two vertices are connected by a path whose internal vertices have distinct colors, and such a path is called a rainbow path. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. Note the trivial fact that $rvc(G) = 0$ if and only if G is a complete graph (here an uncolored graph is also viewed as a colored one with 0 color). Also, clearly, $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. In [5], the authors considered the complexity of determining the rainbow vertex-connection of a graph. In [7] and [8], the authors gave upper bounds for $rvc(G)$ in terms of the minimum degree of G.

For more results on the rainbow connection and rainbow vertex-connection, we refer to the survey [9] and a new book [10] of Li and Sun.

A natural idea is to introduce the concept of strong rainbow vertex-connection. A vertex-colored graph G is *strongly rainbow vertex-connected*, if for every pair u, v of distinct vertices, there exists a rainbow $u - v$ geodesic. The minimum number k for which there exists a k-coloring of G that results in a strongly rainbow vertex-connected graph is called *the strong rainbow vertex-connection number* of G, denoted by $srvc(G)$. Similarly, we have $rvc(G) \leq srvc(G)$ for every nontrivial connected graph G. Furthermore, for a nontrivial connected graph G, we have

$$diam(G) - 1 \leq rvc(G) \leq srvc(G),$$

where $diam(G)$ denotes the diameter of G. The following results on $srvc(G)$ are immediate from definition.

Proposition 1.1 *Let G be a nontrivial connected graph of order n. Then*
(a) $srvc(G) = 0$ *if and only if G is a complete graph;*
(b) $srvc(G) = 1$ *if and only if* $diam(G) = 2$.

Then, it is easy to see the following results.

Corollary 1.2 *Let $K_{s,t}$, $K_{n_1,n_2,\ldots,n_k}$, W_n and P_n denote the complete bipartite graph, complete multipartite graph, wheel and path, respectively. Then*
(1) *For integers s and t with $s \geq 2, t \geq 1$, $srvc(K_{s,t}) = 1$.*
(2) *For $k \geq 3$, $srvc(K_{n_1,n_2,\ldots,n_k}) = 1$.*
(3) *For $n \geq 3$, $srvc(W_n) = 1$.*
(4) *For $n \geq 3$, $srvc(P_n) = n - 2$.*

It is easy to see that if H is a connected spanning subgraph of a nontrivial (connected) graph G, then $rvc(G) \leq rvc(H)$. However, the strong rainbow

vertex-connection number does not have the monotone property. An example is given in Figure 1, where $H = G \setminus v$ is a subgraph of G, but it is easy to check that $srvc(G) = 9 > 8 = srvc(H)$. Here, one has to notice that any two cut vertices must receive distinct colors in a rainbow vertex-coloring, just like a rainbow coloring for which any two cut edges must receive distinct colors.

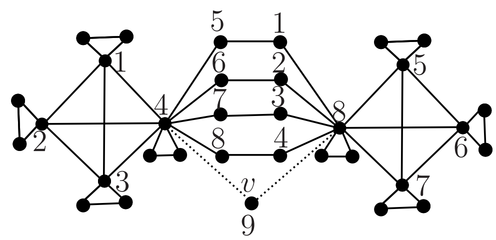

Figure 1.1 A counterexample for the monotonicity of the strong rainbow vertex-connection number.

In this paper, sharp upper and lower bounds of $srvc(G)$ are given for a connected graph G of order n, that is, $0 \leq srvc(G) \leq n-2$. Graphs of order n such that $srvc(G) = 1$, 2, $n-2$ are characterized, respectively. It is also shown that, for each pair a, b of integers with $a \geq 5$ and $b \geq (7a-8)/5$, there exists a connected graph G such that $rvc(G) = a$ and $srvc(G) = b$.

2 Bounds and characterization of extremal graphs

In this section, we give sharp upper and lower bounds of the strong rainbow vertex-connection number of a graph G of order n, that is, $0 \leq srvc(G) \leq n-2$. Furthermore, from the these bounds, we can characterize all the graphs with $srvc(G) = 0, 1, n-2$, respectively. Now we state a useful lemma.

Lemma 2.1 *Let u, v be two vertices of a connected graph G. If the distance $d_G(u,v) \geq diam(G) - 1$, then there exists no geodesic containing both of u and v as its internal vertices.*

Proof. Assume, to the contrary, that there exists a geodesic $R : w_1 - w_2$ containing both u and v as its internal vertices. Then $d_G(w_1, w_2) \geq d_G(u,v) + d_G(w_1, u) + d_G(v, w_2) \geq diam(G) + 1$, which contradicts to the definition of diameter. ∎

Theorem 2.2 *Let G be a connected graph of order n $(n \geq 3)$. Then $0 \leq srvc(G) \leq n-2$. Moreover, the bounds are sharp.*

Proof. For $n = 3$, we know $G = K_3$ or P_3. Since $srvc(K_3) = 0 < n-2$ and $srvc(P_3) = 1 = n-2$, the result holds. Assume $n \geq 4$. If $diam(G) = 1$, then

G is a complete graph and $srvc(G) = 0 \leq n-2$. If $diam(G) = 2$, then we have $srvc(G) = 1 \leq n-2$ by Proposition 1.1.

Now suppose that $diam(G) \geq 3$. Let $diam(G) = k$ and u, v be two vertices at distance k. Let $P : u(= x_0), x_1, x_2, \ldots, x_k(= v)$ be a geodesic connecting u and v. Let c be a $(n-2)$-vertex-coloring of G defined as: $c(u) = c(x_{k-1}) = 1$, $c(x_1) = c(v) = 2$, and assigning the $n-4$ distinct colors $\{3, 4, \ldots, n-2\}$ to the remaining $n-4$ vertices of G. Then we will show that the coloring c is indeed a strong rainbow $(n-2)$-vertex-coloring.

It is easy to see that $d_G(u, x_{k-1}) \geq k-1$ and $d_G(v, x_1) \geq k-1$. From Lemma 2.1, there exists no geodesic containing both of u and x_{k-1} as its internal vertices. The same is true for vertices v and x_1. So, any geodesic connecting any two vertices of G must be rainbow. Thus, we have $0 \leq srvc(G) \leq n-2$.

We show that the bounds are sharp. The complete graph K_n attains the lower bound and the path graph P_n attains the upper bound. ∎

From Theorem 2.2, we know that P_n is the graph satisfying that $srvc(P_n) = n-2$. Actually, P_n is the unique graph with this property. That is the following theorem, which can be easily deduced from Lemma 2.4.

Theorem 2.3 (1) *$srvc(G) = 0$ if and only if G is a complete graph;*

(2) *$srvc(G) = 1$ if and only if $diam(G) = 2$;*

(3) *$srvc(G) = n-2$ if and only if G is a path of order n.* ∎

Lemma 2.4 *Let G be a connected graph of order n ($n \geq 3$). If G is not a path, then $srvc(G) \leq n-3$.*

Proof. For $n = 3$, we know that $G = K_3$ and $srvc(G) = 0 = n-3$. For $n \geq 4$, we distinguish the following two cases according to the minimum degree $\delta(G)$ of G. Let $diam(G) = k$.

Case 1. $\delta(G) \geq 2$.

For $4 \leq n \leq 5$, $srvc(G) \leq 1 \leq n-3$ since $k \leq 2$. Assume $n \geq 6$. If $k = 1$, then G is a complete graph and $srvc(G) = 0 \leq n-3$. If $k = 2$, then it follows that $srvc(G) = 1 \leq n-3$.

Now suppose that $k \geq 3$ and let $P : u(= x_0), x_1, x_2, \ldots, x_k(= v)$ be a geodesic of order k. Since $\delta(G) \geq 2$, there exist two vertices $u'(\neq x_1)$ and $v' \neq x_{k-1}$ such that u' and v' are adjacent to u and v, respectively.

We check whether there exists a geodesic in G containing both of u' and x_{k-1} as its internal vertices or containing both of x_1 and v' as its internal vertices. If G has such geodesics, we choose one, say $Q := w_1$-w_2 containing both of u' and x_{k-1} as its internal vertices. It is easy to see that w_1 and w_2 must be adjacent to u' and x_{k-1}, respectively. We have the following four subcases to consider.

Subcase 1.1. $w_1 \neq u$ and $w_2 = v$.

Since P is a geodesic, $d_G(x_1, v) \geq k-1$. Since $d_G(u, v') + d_G(v', v) \geq d_G(u, v) = k$, we have $d_G(u, v') \geq k-1$. By the same reason, $d_G(u', x_{k-1}) \geq k-2$. Thus $d_G(w_1, x_{k-1}) \geq k-1$. By Lemma 2.1, there exists no geodesic

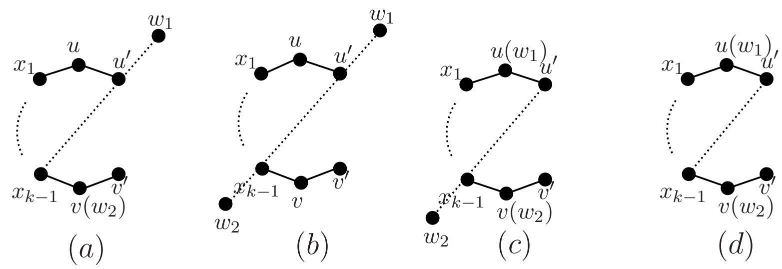

Figure 2.1 Four cases of w_1 and w_2

containing both of w_1 and x_{k-1} as its internal vertices. The same is true for v, x_1 or u, v'. Then the following $(n-3)$-vertex-coloring c_1 of G is a strong rainbow vertex-coloring: $c_1(u) = c_1(v') = 1$, $c_1(v) = c_1(x_1) = 2$, $c_1(w_1) = c_1(x_{k-1}) = 3$ and assigning the $n-6$ distinct colors $\{4, 5, \ldots, n-3\}$ to the remaining $n-6$ vertices. So $srvc(G) \leq n-3$.

Subcase 1.2. $w_1 \neq u$ and $w_2 \neq v$.

It is obvious that $d_G(w_1, x_{k-1}) \geq k-1$ and $d_G(w_2, u') \geq k-1$ and $d_G(u, v) > k-1$. Then the following $(n-3)$-vertex-coloring c_2 of G is a strong rainbow vertex-coloring: $c_2(u') = c_2(w_2) = 1$, $c_2(w_1) = c_2(x_{k-1}) = 2$, $c_2(u) = c_2(v) = 3$ and assigning the $n-6$ distinct colors $\{4, 5, \ldots, n-3\}$ to the remaining $n-6$ vertices. So $srvc(G) \leq n-3$.

Subcase 1.3. $w_1 = u$ and $w_2 \neq v$.

It is easy to see that $d_G(u, x_{k-1}) \geq k-1$ and $d_G(w_2, x_1) \geq k-1$ and $d_G(u', v) \geq k-1$. From Lemma 2.1, we know that the following $(n-3)$-vertex-coloring c_3 of G is a strong rainbow vertex-coloring: $c_3(u') = c_3(v) = 1$, $c_3(u) = c_3(x_{k-1}) = 2$, $c_3(x_1) = c_3(w_2) = 3$ and assigning the $n-6$ distinct colors $\{4, 5, \ldots, n-3\}$ to the remaining $n-6$ vertices. Hence, we have $srvc(G) \leq n-3$.

Subcase 1.4. $w_1 = u$ and $w_2 = v$.

We will show that the following $(n-3)$-vertex-coloring c_4 of G is a strong rainbow vertex-coloring: $c_4(u) = c_4(v) = 1$, $c_4(u') = c_4(x_{k-1}) = 2$, $c_4(x_1) = c_4(v') = 3$ and assigning the $n-6$ distinct colors $\{4, 5, \ldots, n-3\}$ to the remaining $n-6$ vertices.

In this case, we can use the geodesic $P : w_1(= u), x_1, x_2, \ldots, x_{k-1}, v(= w_2)$ instead of geodesic Q to connect w_1 and w_2, which implies that u' and x_{k-1} can be assigned with the same color. From this together with $d_G(u, v) = k$, we know that if there exists no geodesic containing x_1 and v' as its internal vertices, then c_4 is a strong rainbow vertex-coloring.

If there exists a geodesic $R : s_1 - s_2$ containing both of x_1 and v' as its internal vertices and $s_1 \neq u$, $s_2 \neq v$ or $s_1 \neq u$, $s_2 = v$ or $s_1 = u$, $s_2 \neq v$, we can employ similar discussions as the above three subcases of Case 1 to get $srvc(G) \leq n-3$.

For the remaining case that $s_1 = u$ and $s_2 = v$, we can use the geodesic P instead of geodesic R to connect s_1 and s_2, which implies that v' and x_1 can be assigned with the same color. Thus c_4 is indeed a strong rainbow $(n-3)$-vertex-coloring of G, which results in $srvc(G) \leq n-3$.

If there is no geodesic containing both of u' and x_{k-1} as its internal vertices and containing both of v' and x_1 as its internal vertices, then it is obvious that c_4 is also a strong rainbow vertex-coloring of G. Therefore, $srvc(G) \leq n-3$.

Case 2. G has pendant vertices.

In this case, we will show that $srvc(G) \leq n-3$ by induction on n. If $n = 4$, then G must be the star $K_{1,3}$ or a graph obtained by identifying a vertex of K_3 to a vertex of K_2. From Proposition 1.1, we have $srvc(G) = 1 = n-3$ since $diam(G) = 2$. Suppose that the assertion holds for a graph G of smaller order. We can always find a pendant vertex v in G such that $H = G - v$ is not a path. Let u be adjacent to v in G. We distinguish the following two subcases.

Subcase 2.1. $\delta(H) = 1$.

Since H has pendant vertices but H is not a path, by induction hypothesis, $srvc(H) \leq n-4$. We give G a strong rainbow $(n-4)$-vertex-coloring. Without loss of generality, suppose that color 1 was assigned to u in H. Then we give u a fresh color instead of 1 and color v with 1. Such a $(n-3)$-vertex-coloring of G is a strong rainbow vertex-coloring. Thus, we have $srvc(G) \leq n-3$.

Subcase 2.2. $\delta(H) \geq 2$.

In this subcase, we can get $srvc(H) \leq n-4$ by a similar discussion to Case 1. We also can obtain $srvc(G) \leq n-3$ by giving a same vertex-coloring of G as Subcase 2.1.

From the above arguments, we obtain that $srvc(G) \leq n-3$ if G is not a path. ∎

3 The difference of $rvc(G)$ and $srvc(G)$

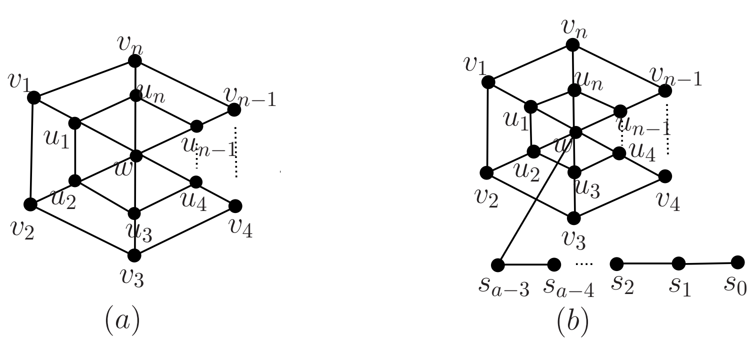

Figure 3.1 Graphs that are used in the proof of Theorem 3.1.

In [4], the authors proved that for any pair a, b of integers with $a = b$ or $3 \leq a < b$ and $b \geq (5a-6)/3$, there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$. Later, Chen and Li [6] confirmed a conjecture of [4] that for any pair a, b of integers, there is a connected graph G such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$. For the two vertex-version parameters $rvc(G)$ and $srvc(G)$, we can obtain a similar result as follows.

Theorem 3.1 *Let a and b be integers with $a \geq 5$ and $b \geq (7a-8)/5$. Then there exists a connected graph G (as shown in Figure 3(b)) such that $rvc(G) = a$ and $srvc(G) = b$.* ∎

We construct a two-layers-wheel graph, denoted by W_n^2, as follows: given two n-cycles $C_n^1 : u_1, u_2, \ldots, u_n, u_1$ and $C_n^2 : v_1, v_2, \ldots, v_n, v_1$, for $1 \leq i \leq n$, join u_i to a new vertex w and v_i (see Figure 3 (a)).

Lemma 3.2 *For $n \geq 3$, the rainbow vertex-connection number of the two-layers-wheel W_n^2 is*

$$rvc(W_n^2) = \begin{cases} 1 & if\ n = 3, \\ 2 & if\ 4 \leq n \leq 6, \\ 3 & if\ 7 \leq n \leq 10, \\ 4 & if\ n \geq 11. \end{cases}$$

Proof. Since $diam(W_3^2) = 2$, it follows that $rvc(W_3^2) = 1$. For $4 \leq n \leq 6$, $diam(W_n^2) = 3$ and then $rvc(W_n^2) \geq 2$. Given a 2-coloring c_1 as follows: $c_1(w) = 2$, $c_1(u_i) = 1$ for $1 \leq i \leq n$, $c_1(v_i) = 1$ when i is odd and $c_1(v_i) = 2$ otherwise. Observe that c_1 is a rainbow vertex-coloring, which implies that $rvc(W_n^2) = 2$ for $4 \leq n \leq 6$.

For $n = 7$, $diam(W_n^2) = 3$ and so $rvc(W_n^2) \geq 2$. We will show that $rvc(W_n^2) \neq 2$. Assume, to the contrary, that $rvc(W_n^2) = 2$. Let c' be a rainbow 2-coloring of W_n^2. We consider the rainbow path connecting v_1 and v_4. Since $rvc(W_n^2) = 2$, v_2 and v_3 must have distinct colors. Without loss of generality, let $c'(v_2) = 1$ and $c'(v_3) = 2$. Similarly, v_6 and v_7 must have distinct colors if considering the rainbow path connecting v_1 and v_5. If $c'(v_6) = 1$ and $c'(v_7) = 2$, then there is no rainbow path connecting v_2 and v_6 if $c'(v_1) = 2$ and also no rainbow path connecting v_3 and v_7 if $c'(v_1) = 1$. Now suppose $c'(v_6) = 2$ and $c'(v_7) = 1$, there is no rainbow path connecting v_2 and v_6 if $c'(v_1) = 1$. Thus $c'(v_1) = 2$. By the same reason, v_4 and v_5 must have distinct colors. But there is no rainbow path connecting v_4 and v_7 if $c'(v_4) = 1$ and $c'(v_5) = 2$ and no rainbow path connecting v_2 and v_5 if $c'(v_4) = 2$ and $c'(v_5) = 1$, a contradiction. Hence, we have $rvc(W_n^2) \geq 3$. Define the 3-coloring c_2 of W_n^2 as follows: $c_2(w) = 3$, $c_2(u_i) = 1$ when i is odd and $c_2(u_i) = 2$ otherwise; $c_2(v_i) = 3$ when i is odd and $c_2(v_i) = 2$ when i is even for $1 \leq i \leq 5$, $c_2(v_6) = 1$, $c_2(v_7) = 2$. It is easy to check that c_2 is a rainbow vertex-coloring, which means that $rvc(W_n^2) = 3$ for $n = 7$.

For $8 \leq n \leq 9$, $diam(W_n^2) = 4$ and so $rvc(W_n^2) \geq 3$. In this case, we define the 3-coloring c_3 of W_n^2 as follows: $c_3(w) = 3$, $c_3(u_i) = 1$ when i is odd and $c_3(u_i) = 2$ otherwise, $c_3(v_i) = 1$ when $i \equiv 2(mod\ 3)$, $c_3(v_i) = 2$ when $i \equiv 0(mod\ 3)$, $c_3(v_i) = 3$ when $i \equiv 1(mod\ 3)$. This coloring is also a rainbow vertex-coloring and it follows that $rvc(W_n^2) = 3$ for $8 \leq n \leq 9$.

For $n = 10$, $diam(W_n^2) = 4$ and so $rvc(W_n^2) \geq 3$. Define the 3-coloring c_4 as follows: $c_4(w) = 3$, $c_4(u_i) = 1$ for $1 \leq i \leq 5$ and $c_4(u_i) = 2$ for $6 \leq i \leq 10$, $c_4(v_1) = 2$, $c_4(v_i) = i-1$ for $2 \leq i \leq 4$, $c_4(v_i) = i-4$ for $5 \leq i \leq 7$, $c_4(v_8) = 2$, $c_4(v_9) = 1$, $c_4(v_{10}) = 3$. One can check that c_4 is a rainbow vertex-coloring and it follows that $rvc(W_n^2) = 3$ for $n = 10$.

Finally, suppose that $n \geq 11$. Observe that the 4-coloring c is a rainbow vertex-coloring: $c(w) = 3$, $c(u_i) = 1$ when i is odd and $c(u_i) = 2$ otherwise, $c(v_i) = 4$ for all i. It remains to show that $rvc(W_n^2) \geq 4$. Assume, to the contrary, that $rvc(W_n^2) = 3$. Let c' be a rainbow 3-vertex-coloring of W_n^2. Without loss of generality, assume that $c'(u_1) = 1$. For each i with $6 \leq i \leq n-4$, v_1, u_1, w, u_i, v_i is the only $v_1 - v_i$ path of length 4 in W_n^2 and so u_1, w and u_i must have different colors. Without loss of generality, let $c'(w) = 3$ and $c'(u_i) = 2$. Since $c'(u_6) = 2$, it follows that $c'(u_n) = 1$. This forces $c'(u_5) = 2$, which in turn forces $c'(u_{n-1}) = 1$. Similarly, $c'(u_{n-1}) = 1$ forces $c'(u_4) = 2$; $c'(u_4) = 2$ forces $c'(u_{n-2}) = 1$; $c'(u_{n-2}) = 1$ forces $c'(u_3) = 2$; $c'(u_3) = 2$ forces $c'(u_{n-3}) = 1$; $c'(u_{n-3}) = 1$ forces $c'(u_2) = 2$. There is no rainbow $v_2 - v_7$ path in W_n^2, which is a contradiction. Therefore, $rvc(W_n^2) = 4$ for $n \geq 11$. ∎

Lemma 3.3 *For $n \geq 3$, the strong rainbow vertex-connection number of the two-layers-wheel W_n^2 is*

$$srvc(W_n^2) = \begin{cases} \lceil \frac{n}{5} \rceil & if\ n = 3, 6; \\ \lceil \frac{n}{5} \rceil + 1 & if\ n \geq 4\ and\ n \neq 6. \end{cases}$$

Proof. Since $diam(W_3^2) = 2$, it follows by Proposition 1.1 that $srvc(W_3^2) = 1$. If $4 \leq n \leq 6$, we can check that the coloring c_1 given in the proof of Lemma 3.2 is a strong rainbow 2-vertex-coloring. So $srvc(W_n^2) \leq 2$. From this together with $srvc(W_n^2) \geq rvc(W_n^2) = 2$, it follows that $srvc(W_n^2) = 2$. If $7 \leq n \leq 10$, we can check that the coloring c_2, c_3 and c_4 given in the proof of Lemma 3.2 is a strong rainbow 3-vertex-coloring. So $srvc(W_n^2) \leq 3$. Combining this with $srvc(W_n^2) \geq rvc(W_n^2) = 3$, we have $srvc(W_n^2) = 3$.

Now we may assume that $n \geq 11$. Then there is an integer k such that $5k - 4 \leq n \leq 5k$. We first show that $srvc(W_n^2) \geq k+1$. Assume, to the contrary, that $srvc(W_n^2) \leq k$. Let c be a strong rainbow k-vertex-coloring of W_n. If $C_n^1 \cup \{w\}$ uses all the k colors, it is easy to see that w and u_i must have distinct colors, which implies $c(u_j) \in \{1, 2, \ldots, k-1\}$ for $1 \leq i \leq n$. If there exists one color which only appears in $V(C_n^2)$, then we also have $c(u_j) \in \{1, 2, \ldots, k-1\}$ for $1 \leq i \leq n$. Since $d(w) = n > 5(k-1)$, there exists one subset $S \subseteq V(C_n^1)$ such that $|S| = 6$ and all vertices in S are colored the same. Thus, there exist at least two vertices u', $u'' \in S$ such that $d_{C_n^1}(u', u'') \geq 5$ and $d_P(u', u'') = 4$, where $P := v', u', w, u'', v''$. Since P is the

only $u' - u''$ geodesic in W_n^2, it follows that there is no rainbow $v' - v''$ geodesic in W_n^2, a contradiction. Therefore, $srvc(W_n^2) \geq k + 1$.

To show that $srvc(W_n^2) \leq k + 1$, we provide a strong rainbow $(k + 1)$-vertex-coloring c^*: $V(W_n^2) \to \{1, 2, \ldots, k + 1\}$ of W_n^2 defined by

$$c^*(v) = \begin{cases} k+1 & v = w \\ 1, & if\ v = v_i\ and\ i \equiv 2 (mod\ 5), \\ 2 & if\ v = v_i\ and\ i \equiv 3 (mod\ 5), \\ 3 & if\ v = v_i\ and\ i \equiv 4 (mod\ 5), \\ j+1 & if\ v = u_i\ i \in \{5j+1, \ldots, 5j+5\}\ for\ 0 \leq j \leq k-1, \\ 1 & otherwise. \end{cases}$$

Therefore, $srvc(W_n^2) = k + 1 = \lceil \frac{n}{5} \rceil + 1$ for $n \geq 11$. ∎

Proof of Theorem 3.1. Let $n = 5b - 5a + 10$ and let W_n^2 be the two-layers-wheel. Let G be the graph constructed from W_n^2 and the path P_{a-1} : $s_0, s_1, s_2, \ldots, s_{a-2}$ of order $a - 1$ by identifying w and s_{a-2} (see Figure 3 (b)).

First, we show that $rvc(G) = a$. Since $b \geq (7a - 8)/5$ and $a \geq 5$, it follows that $b > a$ and so $n = 5b - 5a + 10 > 11$. By Lemma 3.2, we then have $rvc(G) = 4$. Define a vertex-coloring c of the graph G by

$$c(v) = \begin{cases} 1 & if\ v = v_i\ for\ 1 \leq i \leq n, \\ a & if\ v = u_i\ and\ i\ is\ odd, \\ a-1 & if\ v = u_i\ and\ i\ is\ even, \\ i & if\ v = s_i\ for\ 1 \leq i \leq a-2. \end{cases}$$

It follows that $rvc(G) \leq a$, since c is a rainbow a-vertex-coloring of G.

It remains to show that $rvc(G) \geq a$. Assume, to the contrary, that $rvc(G) \leq a - 1$. Let c' be a rainbow $(a - 1)$-vertex-coloring of G. Since the path $s_0, s_1, s_2, \ldots, s_{a-2}(= w), u_i$ is the only $s_0 - u_i$ path in G, the internal vertices of this path must be colored differently by c'. We may assume, without loss of generality, that $c'(s_i) = i$ for $1 \leq i \leq a - 2$. For each j with $1 \leq j \leq 5b - 5a + 10$, there is a unique $s_0 - v_j$ path of length a in G and so $c'(u_j) = a - 1$ for $1 \leq j \leq 5b - 5a + 10$. Consider the vertices v_1 and v_{a+2}. Since $b \geq (7a - 8)/5$, $n = 5b - 5a + 10 \geq 2a + 2$ and the possible rainbow path connecting v_1 and v_{a+2} must be $v_1, u_1, w, u_{a+2}, v_{a+2}$. But it is impossible since $c'(u_1) = c'(u_{a+2}) = a - 1$, which implies that there is no $v_1 - v_{a+2}$ rainbow path, contradicting our assumption that c' is a rainbow $(a - 1)$-coloring of G. Thus, $rvc(G) \geq a$ and then $rvc(G) = a$.

In the following, we show that $srvc(G) = b$. Since $n = 5b - 5a + 10 = 5(b - a + 2) > 11$, it follows from Lemma 3.3 that $srvc(W_n^2) = b - a + 3$. Let c_1 be a strong rainbow $(b - a + 3)$-vertex-coloring of W_n^2. Define a vertex-coloring c of the graph G by

$$c(v) = \begin{cases} c_1(v) & if\ v \in V(W_n^2); \\ b - a + 3 + i & if\ v = s_i\ for\ 0 \leq i \leq a-3. \end{cases}$$

It follows that $srvc(G) \leq b$, since c is a strong rainbow b-vertex-coloring of G.

It remains to show that $srvc(G) \geq b$. Assume, to the contrary, that $srvc(G) \leq b-1$. Let c^* be a strong rainbow $(b-1)$-vertex-coloring of G. We may assume, without loss of generality, that $c^*(s_i) = i$ for $1 \leq i \leq a-2$. For each j with $1 \leq j \leq 5b-5a+10$, there is a unique $s_0 - w - v_j$ geodesic in G, implying $c^*(u_j) \in \{a-1, a, \ldots, b-1\}$. Let $S = \{u_j : 1 \leq j \leq 5b-5a+10\}$. Then $|S| = 5b-5a+10$ and $|C| = b-a+1$. Since at most five vertices in S can be colored the same, the $b-a+1$ colors in C can color at most $5(b-a+1) = 5b-5a+5$ vertices, producing a contradiction. Therefore, $srvc(G) \geq b$ and so $srvc(G) = b$. ∎

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